Noninvertible symmetry and Topological stability of solitons

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Contents

- Review of Generalized Symmetries in QFT
- Application: Topological stability of soliton beyond homotopy groups (the work with Shi Chen (U. Minnesota), 2210.13780, 2307.00939)

Disclaimer

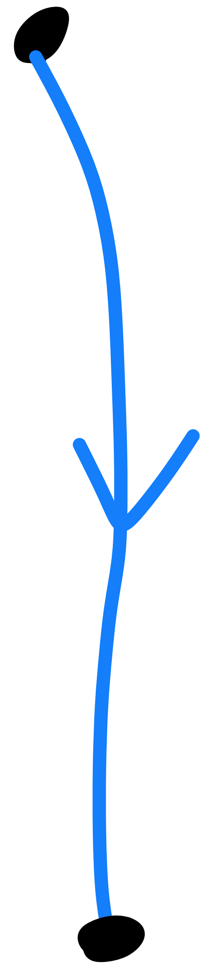
No pheno. application will be discussed in my talk; I'm focusing on basic formal ideas.

Stay tuned to the next talk by Sungwoo Hong !!

Solving Quantum Field Theories

UV $\leftarrow$ microscopic description

- Quantum Chromodynamics
- (Lattice) quantum many-body systems
- Conformal field theories (CFT_{UV})



Renormalization Group (RG) flow

IR $\leftarrow$ macroscopic description

- Nonlinear σ-model (chiral Lagrangian)
- Fermi liquid, spin liquid (TQFT), ...
- CFT_{IR}.

This is a very tough problem!

Power of Symmetry

Sometimes, we can know about low-energy dynamics using Symmetry without solving microscopic Hamiltonian.

e.g. In '60s, people didn't know about Quantum Chromodynamics (QCD).
fundamental theory of strong interaction.

current algebra (chiral effective Lagrangian)

$\Longrightarrow$ successful description of low-energy properties of strong interaction

Why this was possible?

$\boxed{\text{Universality due to SSB of chiral symmetry.}}$

Symmetry in Quantum Mech. & Symmetry in QFT

QM States $|\psi\rangle \in \mathcal{H}$ play the primary role.

- Measure the probability via the Born rule $|\langle i | \psi \rangle|^2$.

Symmetry in QM = Transformation on the states respecting the Born rule.

= (Anti-) Unitary operation on $\mathcal{H}$ (that commutes w/ $\hat{H}$).
(Wigner's thm)

QFT Correlation function $\langle \phi(x_1) \cdots \phi(x_n) \rangle$ play the primary role.

Symmetry in QFT = "Local" operation preserving correlators

Continuous symmetry in QFT : Ward-Takahashi identity

Noether : Assume the Lagrangian $S[\phi]$ is invariant under some continuous transformation $\phi \to e^{i\varepsilon}\phi$.

$\rightsquigarrow$ Local conserved current $J^{\mu}(x)$

$$\partial_{\mu} J^{\mu}(x) = 0 .$$

$$\Big\Downarrow \text{ Quantize}$$

In QFT, this becomes the WT id.

$$\left\langle \partial_{\mu} J^{\mu}(x)\, \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \right\rangle = i \sum_{k=1}^{n} \delta(x-x_k) \left\langle \mathcal{O}_1 \cdots \delta\mathcal{O}_k \cdots \mathcal{O}_n \right\rangle .$$

$\rightsquigarrow$ Various theorems (ex. Selection rule, Nambu-Goldstone thm, Anomaly matching ...)

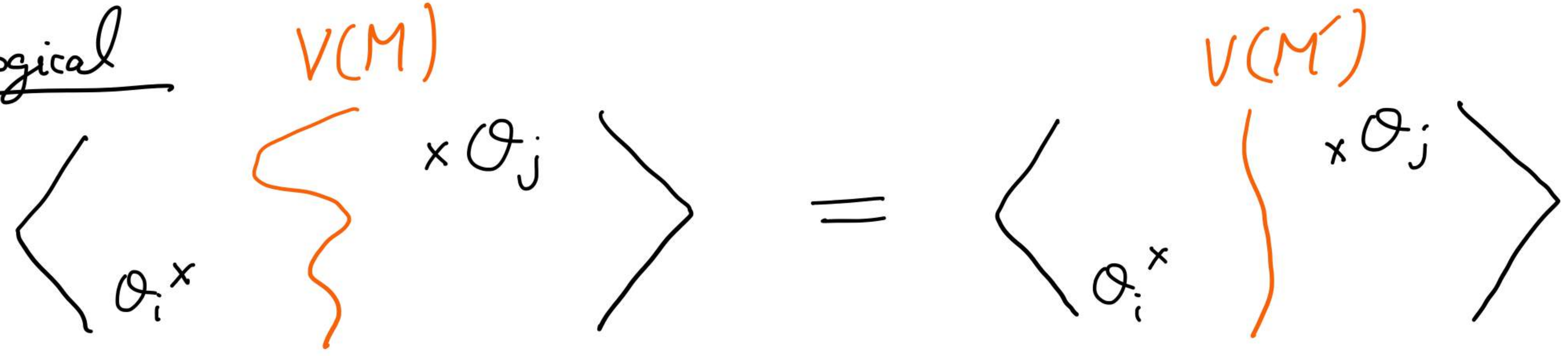
Modern definition of (generalized) symmetry

WT identity gives the basic tool to study QFTs in view of symmetry.
We generalize the notion of symmetry by generalizing the WT identity.

Take-Home Message

$$\text{(Generalized) Symmetry} = \text{Topological operators/defects in QFTs}$$

<u>Topological</u>



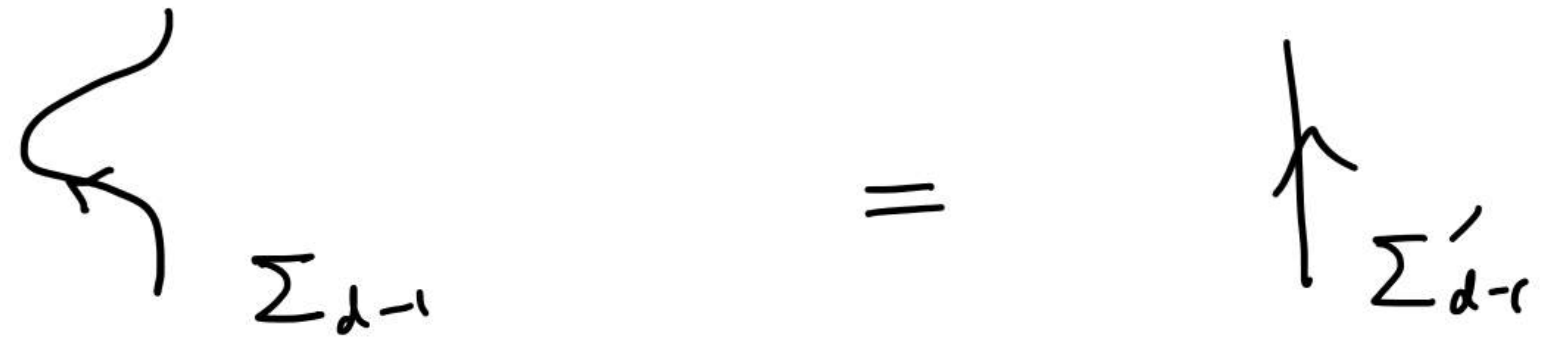
$\Longleftrightarrow$ Conservation law

Ordinary symmetries (or Symmetries in QFT textbooks) in modern viewpoint

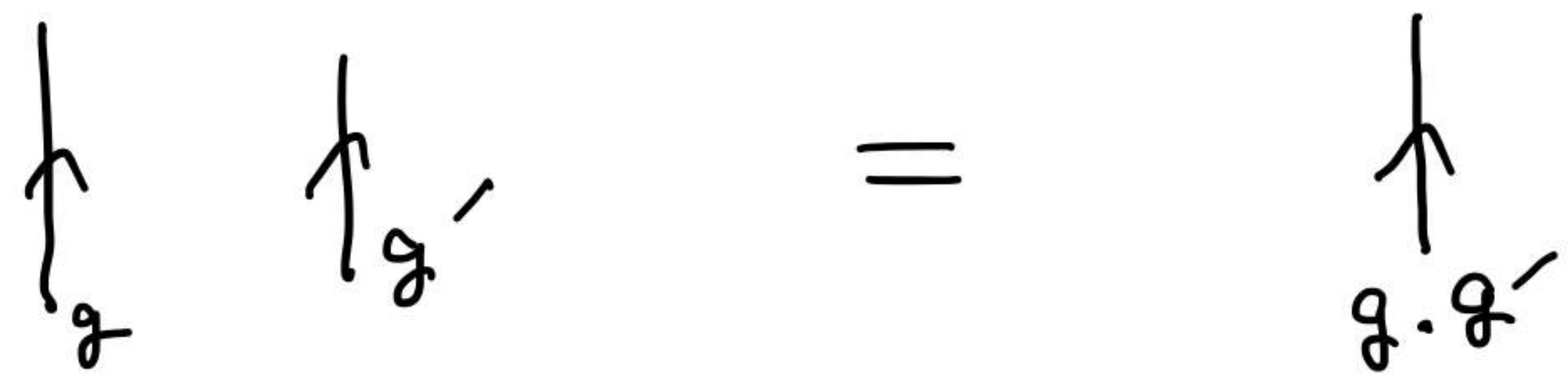
d-dim QFT has symmetry w/ the group G

$\overset{\text{Def}}{\Longleftrightarrow}$

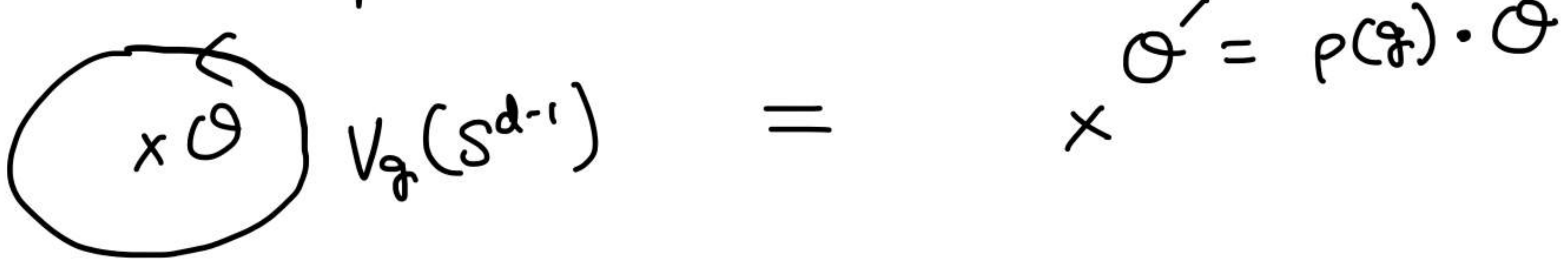
- $^{\exists} V_g(\Sigma_{d-1})$: Topological operator on codim-1 manifolds Σ_{d-1} labelled with $g \in G$.



- Group-like fusion rule : $V_g(\Sigma_{d-1}) \cdot V_{g'}(\Sigma_{d-1}) = V_{g \cdot g'}$



- Local action on point-like operators



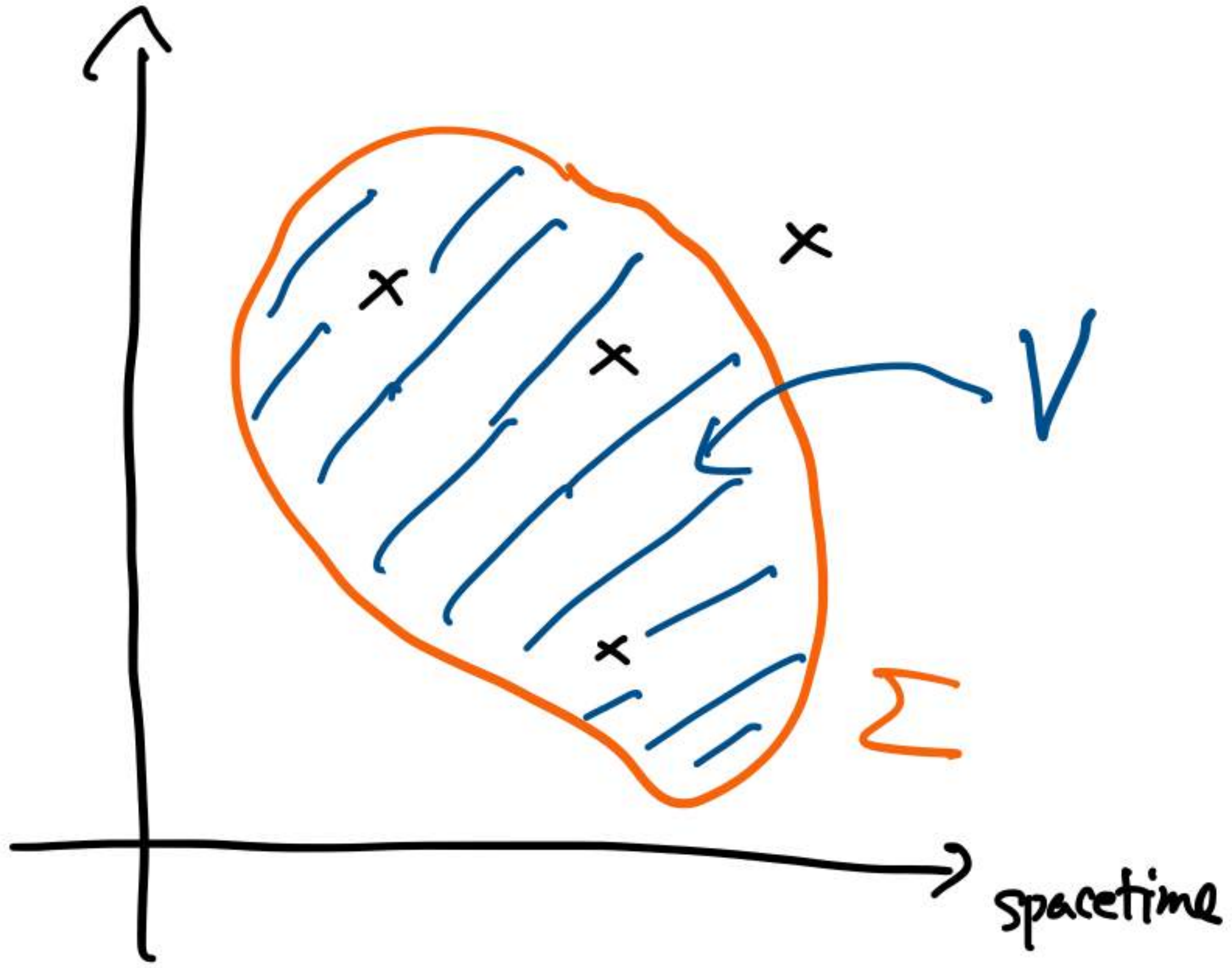
WT revisited

WT identity (in a differential version)

$$\partial_\mu^x \langle J^\mu(x)\, \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = i \sum_{k=1}^{n} \delta(x - x_k) \langle \mathcal{O}_1(x_1) \cdots \delta\mathcal{O}_k(x_k) \cdots \mathcal{O}_n \rangle$$

$\Downarrow$ $\int_V d^dx$ on both sides (with $\partial V = \Sigma$)

$\Uparrow$ Take small variations for Σ.



WT identity (in an integrated version)

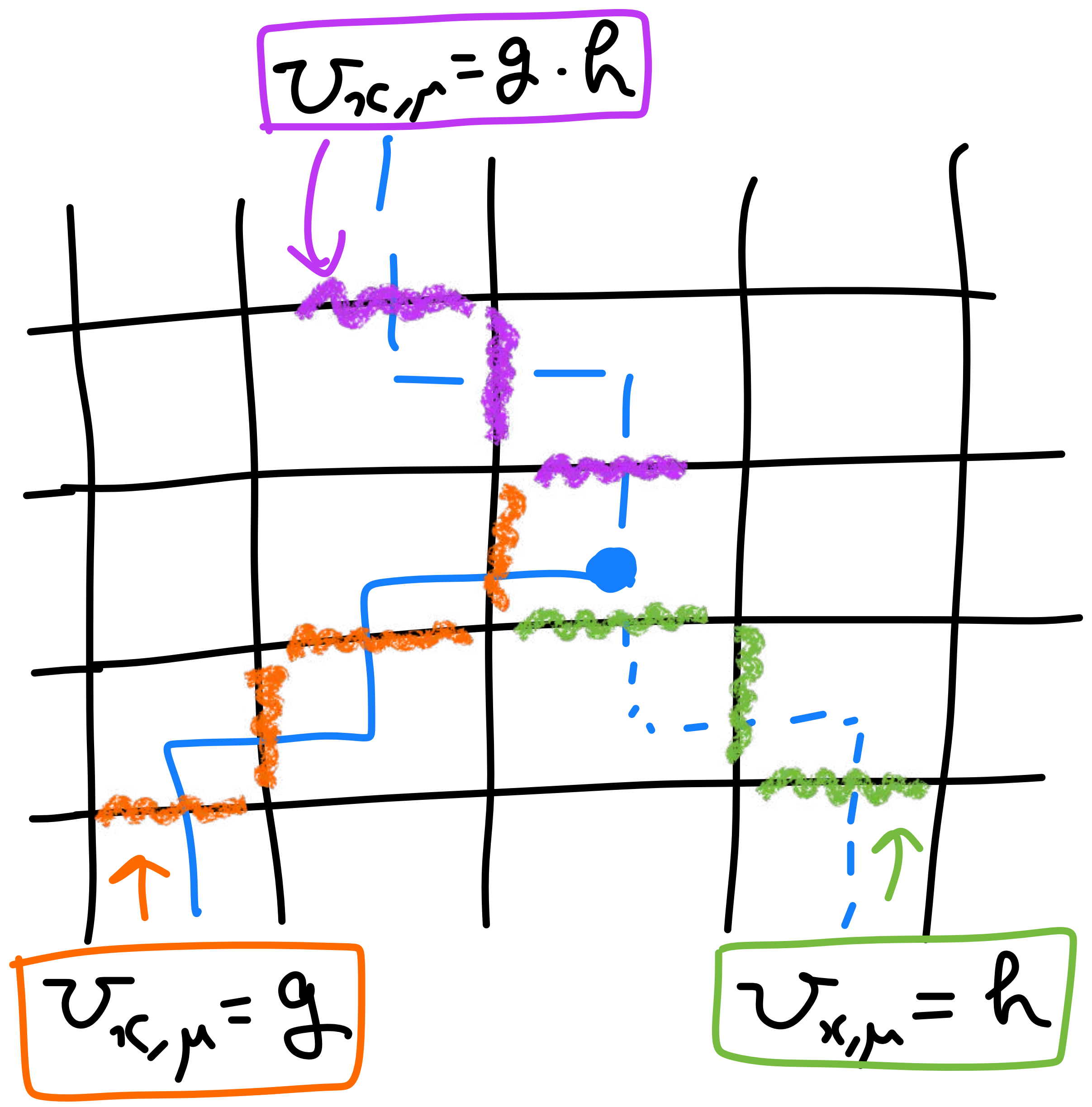
$$\Big\langle \underbrace{\Big(\int_\Sigma J^\mu dS_\mu\Big)}\, \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \Big\rangle = i \sum_{x_k \in V} \langle \mathcal{O}_1 \cdots \delta\mathcal{O}_k \cdots \mathcal{O}_n \rangle$$

$$V_{g=e^{i\varepsilon}}(\Sigma) = 1 + i\varepsilon \int_\Sigma J^\mu dS_\mu + \cdots \quad \text{Topological operator}$$

Network of topological defects

Model $$S[\phi] = K\sum_{x,\mu}\left(\phi^\dagger_{x+\mu}\cdot\phi_x + \phi^\dagger_x\cdot\phi_{x+\mu}\right) + V[\phi] \qquad (\phi_x\in\mathbb{C}^N)$$

Assume that G is a symmetry (For $g\in G$, $\phi_x \mapsto R(g)\cdot\phi_x$ w/ $R(g)\in U(N)$).



$\Longleftrightarrow$

$$S_{twist} = K\sum_{x,\mu}\left(\phi^\dagger_{x+\mu}\, R(U_{x,\mu})\,\phi_x + \phi^\dagger_x\cdot R(U_{x,\mu})^\dagger\cdot\phi_{x+\mu}\right) + V[\phi]$$

with the twisted hopping (i.e. "link variable")

$U_{x,\mu}\in G$.

(Network of topological defects (consistent with fusion rule))

(Background G gauge field (with flatness condition))

(This is the lattice definition of $\exp\left(i\epsilon^a\int_\Sigma J^\mu_a(x)\,dS_\mu\right)$.)

Various direction of generalizations

Conventional symmetry $\longrightarrow$ **Higher-form / Higher-group symmetry**

Conventional symmetry:
- Codim-1 topological operator
- Group-like fusion rule

Higher-form / Higher-group symmetry:
- Topological operators w/ higher codimensions.
- Charged operators are extended objects
- Fusion rule is invertible.

[Gaiotto, Kapustin, Seiberg, Willet '14, ...]
[Sharpe '15, Cordova, Dumitrescu, Intrilligator '17, YT, Ünsal '19, Hidaka, Nitta, Yokokura '20, ...]

$\downarrow$

Noninvertible / Categorical symmetry

- Non-group-like fusion rule.
  "Inverse" does not necessarily exist : Noninvertible symmetry
- Math : Group $\Rightarrow$ (Higher) Category : Categorical symmetry

[2d : Verlinde '88, Bhardwaj, Tachikawa '17, Chang, Lin, Shao, Wang, Yin '18, Thorngren, Wang '19]
[≥3d : Nguyen, YT, Ünsal '21, Koide, Nagoya, Yamaguchi '21, Choi, Cordova, Hsin, Lam, Shao '21, Kaidi, Ohmori, Zheng '21, ...]

Noninvertible symmetry

Conventionally, symmetry forms a group. $\left(V_g(\Sigma) V_{g'}(\Sigma) = V_{gg'}(\Sigma) \right)$.

Is this necessary? or just an option?

Our motto : $\boxed{\text{Symmetry} = \text{Topological operators in QFT.}}$

$\Rightarrow$ More general fusion rule is allowed

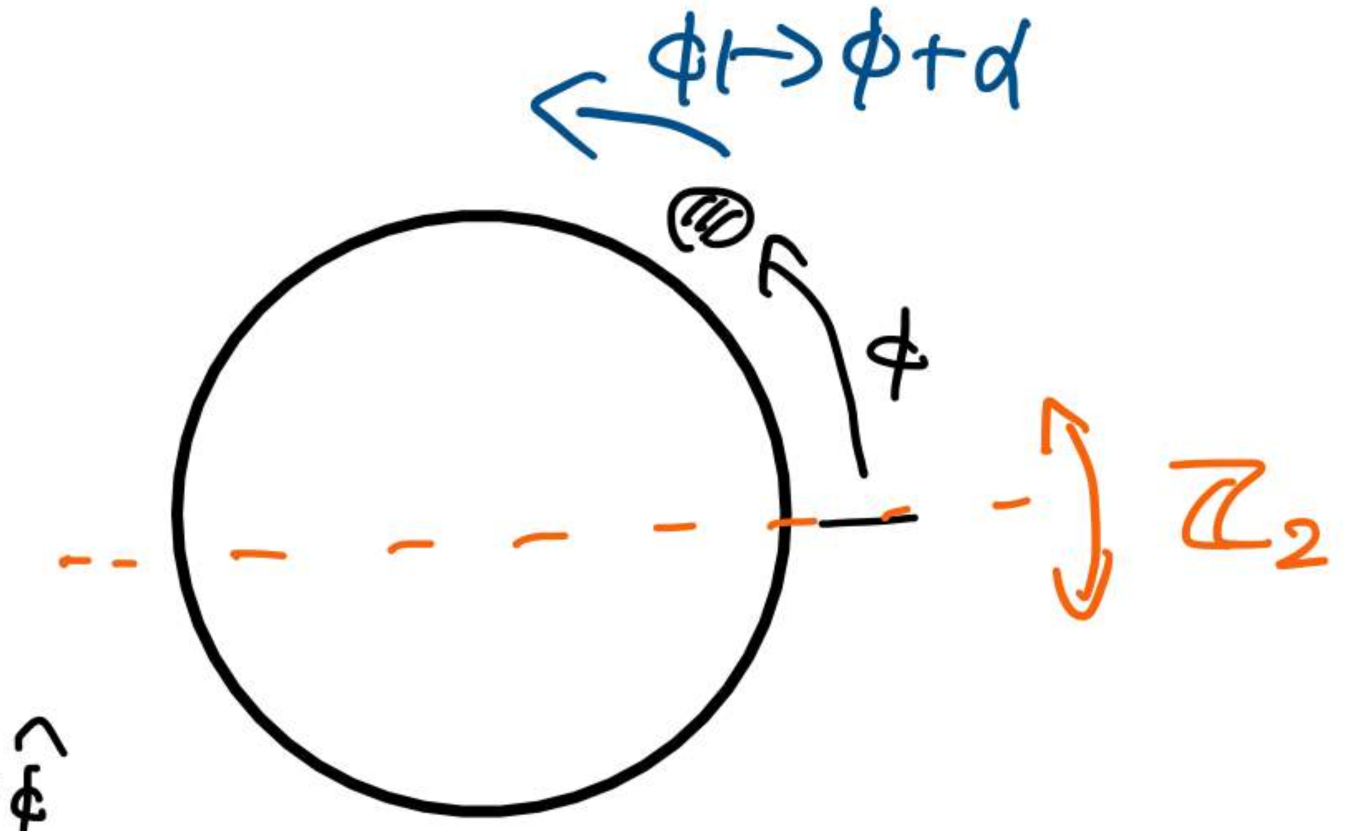
$$V_a(\Sigma)\, V_b(\Sigma) = \sum_c N_{ab}^c(\Sigma)\, V_c(\Sigma).$$

Nontrivial constraints arise by the locality of QFT,

but the non-invertible fusion rule is possible.

Toy example : Particle on $S^1/\mathbb{Z}_2$

$$S = \frac{m}{2}\int d\tau \left(\frac{d\phi}{d\tau}\right)^2 \quad \text{w/} \quad \phi \sim \phi + 2\pi .$$



This has the symmetry $O(2) \cong \underbrace{U(1)}_{\phi \mapsto \phi+\alpha} \rtimes \underbrace{\mathbb{Z}_2}_{\phi \mapsto -\phi}$.

$$\hat{U}_\alpha = e^{i\alpha\hat{p}} \quad : \quad \hat{U}_\alpha e^{i\hat{\phi}} \hat{U}_\alpha^{-1} = e^{i\alpha} e^{i\hat{\phi}}$$

$$\hat{C} \quad : \quad \hat{C} e^{i\hat{\phi}} \hat{C}^{-1} = e^{-i\hat{\phi}}, \quad \hat{C}\hat{p}\hat{C}^{-1} = -\hat{p}.$$

$\Downarrow$ Gauge $\hat{C}$ (i.e. restrict the Hilbert space to $\hat{C}$-even sector)

$\hat{U}_\alpha$ is not $\hat{C}$-invariant (except $\alpha = 0, \pi$). The gauge-inv. object is

$$\hat{V}_\alpha = \hat{U}_\alpha + \hat{U}_{-\alpha} = 2\cos(\alpha\hat{p}).$$

Non-group-like fusion rule: $\hat{V}_\alpha \hat{V}_\beta = \hat{V}_{\alpha+\beta} + \hat{V}_{\alpha-\beta}$.

Non-invertible 1-form symmetry

(Nguyen, YT, Ünsal ; Heidenreich, McNamara, Montero, Reece, Rudelius, Valenzuela, ...)

Pure $U(1)$ gauge theory has $U(1)^{[1]}$ and $(\mathbb{Z}_2)_C$ (charge conjugation).

$\xRightarrow{\text{Gauging } (\mathbb{Z}_2)_C}$ Gauge group becomes $\underline{O(2) = U(1) \rtimes \mathbb{Z}_2}$.

↑ Center of $O(2)$ is just $\mathbb{Z}_2$, which is much smaller than $U(1)$.

Do we lose most part of 1-form symmetry by gauging $(\mathbb{Z}_2)_C$?

A. They survive as non-invertible symmetry!

$U_\alpha(M_{d-2})$: $U(1)$ 1-form symmetry generator in $U(1)$ gauge theory

$\Downarrow$ $(\mathbb{Z}_2)_C$ gauging

$U_\pi(M_{d-2})$: $\mathbb{Z}_2^{[1]}$ defect for $O(2)$ gauge theory

$U'_\alpha(M_{d-2}) = U_\alpha + U_{-\alpha}$: Non-invertible 1-form symmetry generators of $O(2)$ gauge theory.

$$\left(U'_\alpha U'_\beta = U'_{\alpha+\beta} + U'_{\alpha-\beta} \right)$$

[In 3d $O(2)$ gauge theory with monopoles, selection rule of confining strings obey this non-inv. symmetry.]

Another example : Kramers - Wannier duality symmetry

2d Ising CFT $\xleftrightarrow{\text{Jordan-Wigner}}$ 2d massless Majorana fermion

$$\mathcal{L} = \psi^T \gamma^\mu \partial_\mu \psi$$
$$= \psi_- \partial_+ \psi_- + \psi_+ \partial_- \psi_+ .$$

dual symmetry

Symmetry

$\boxed{\mathbb{Z}_2}$: $S(x) \to -S(x)$

Symmetry

$\boxed{(\mathbb{Z}_2)_F} \times \boxed{(\mathbb{Z}_2)_\chi}$

$\psi_\pm \mapsto -\psi_\pm$

$\begin{cases} \psi_+ \to -\psi_+ \\ \psi_- \to \psi_- \end{cases}$

The bosonic counterpart of the chiral symmetry $(\mathbb{Z}_2)_\chi$ is missing within the conventional symmetries:

It is realized as a noninvertible symmetry, ie. Kramers-Wannier duality.

Details on Kramers-Wannier duality

$$\text{Ising at } \beta \;=\; \text{Ising}/\underbrace{\mathbb{Z}_2}_{\text{Gauging } \mathbb{Z}_2 \text{ symmetry}} \text{ at } \tilde{\beta}$$

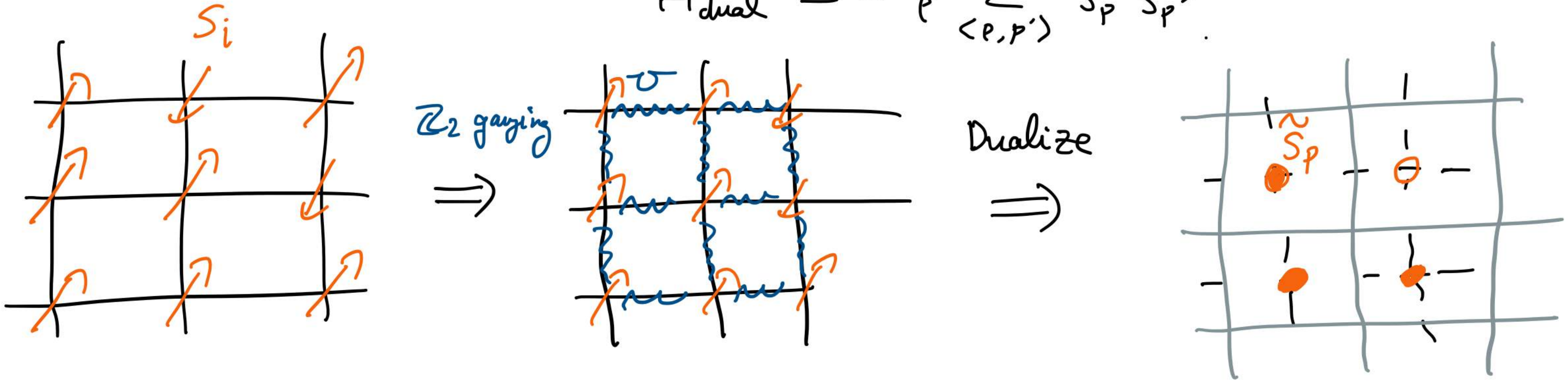
$$H = -\beta \sum_{\langle i,j\rangle} s_i s_j \;\xRightarrow[\mathbb{Z}_2 \text{ gauge field}]{\text{Introduce the flat}}\; H_{\text{gauged}} = -\beta \sum_{\langle i,j\rangle} s_i \, \upsilon_{ij} \, s_j \quad \text{w/ } \upsilon_{ij} \in \{\pm 1\}$$

Flatness condition:

$$\delta_{\prod_{\langle i,j\rangle \in p} \upsilon_{ij},\, 1} = \frac{1}{2} \sum_{\tilde{s}_p = \pm 1} \Big(\prod_{\langle i,j\rangle \in p} \upsilon_{ij} \Big)^{\frac{1-\tilde{s}_p}{2}}$$

$\xRightarrow[s_i \,\&\, \upsilon_{ij}]{\text{Summing over}}$ Ising model described by dual spins $\tilde{s}_p$

$$H_{\text{dual}} = -\tilde{\beta} \sum_{\langle p,p'\rangle} \tilde{s}_p \tilde{s}_{p'} .$$

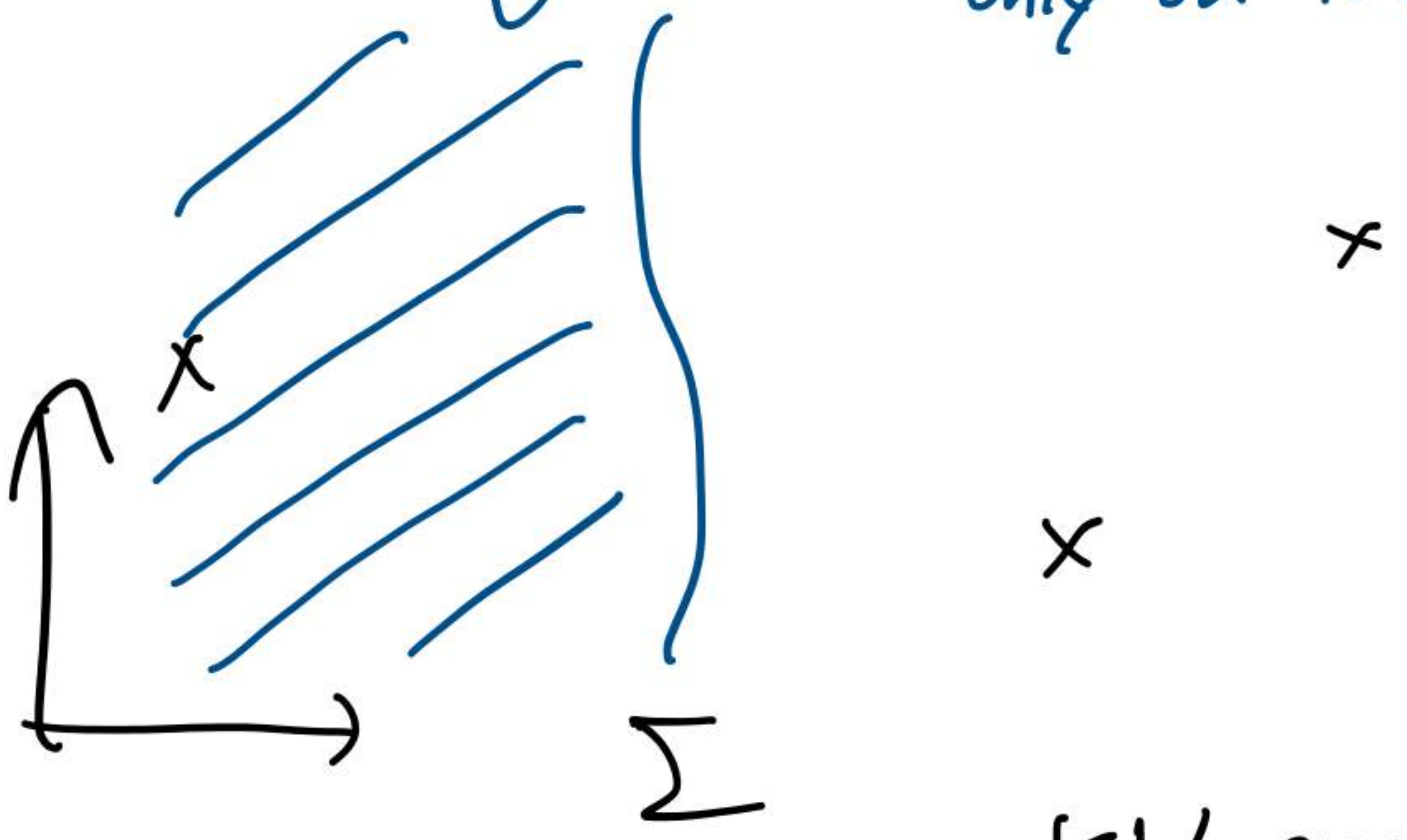


Duality & Half-space gauging (cf. Aasen-Fendley-Mong, Koide, Nagoya, Yamaguchi, Choi, Cordova, Hsin, Lam, Shao, Kaidi, Ohmori, Zhen, ...)

At the self-dual temperature $\beta_c = \beta = \tilde{\beta}(\beta)$,

there exists the codim-1 topological operator associated w/ KW duality

Gauge $\mathbb{Z}_2$ & Apply duality only on the half of the spacetime



$$= \left\langle \mathcal{D}(\Sigma) \; \mathcal{O}_1 \cdots \mathcal{O}_n \right\rangle$$

KW symmetry

$\mathbb{Z}_2$ symmetry

$$\mathcal{D}(\Sigma) \times \mathcal{D}(\Sigma) = 1 + \eta(\Sigma).$$

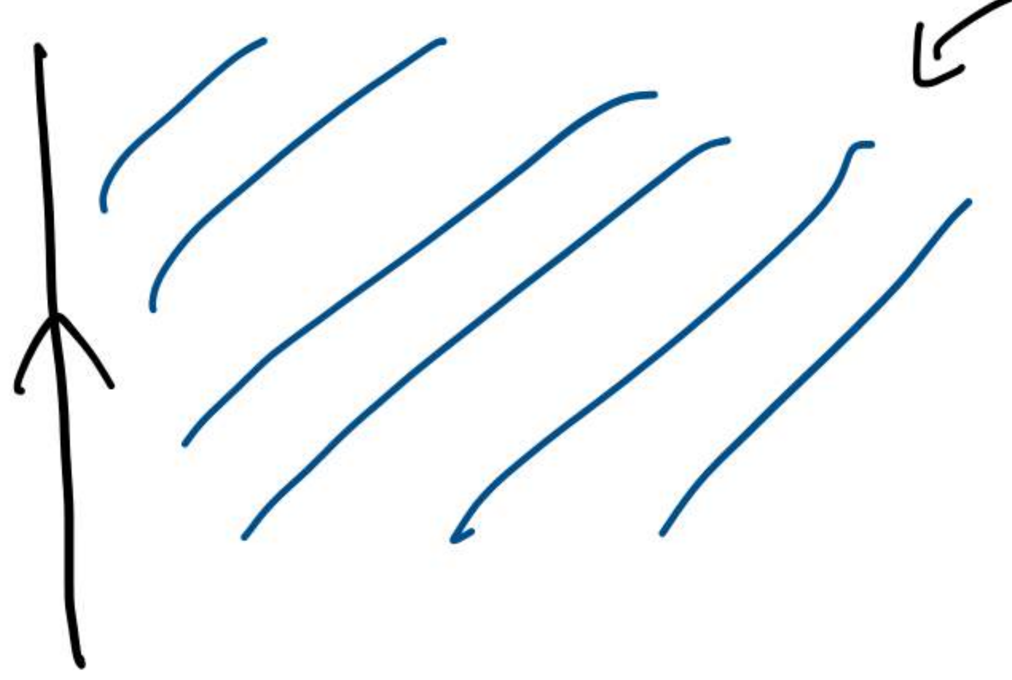
What is the general rule ?

In many cases, the construction process of noninvertible symmetries is nontrivial.

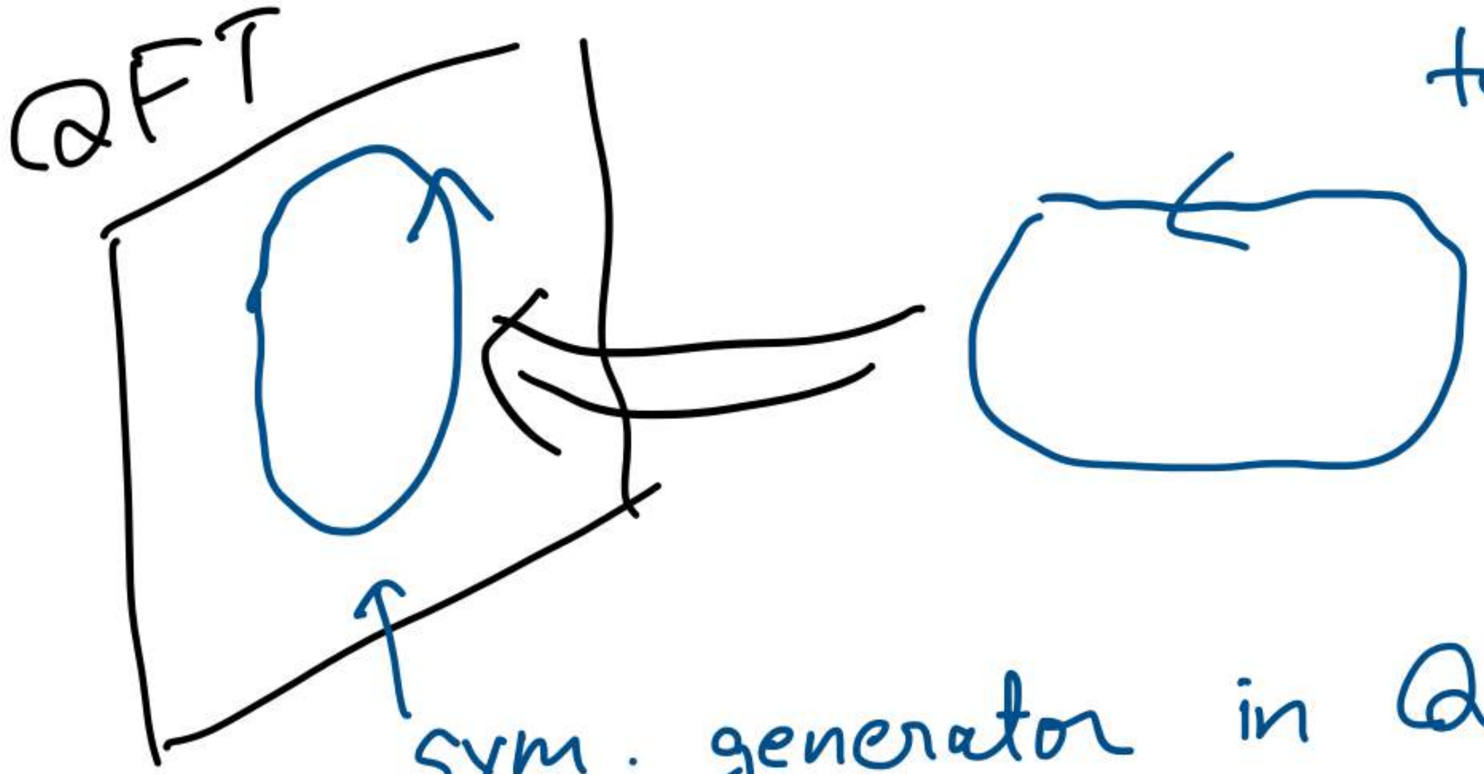
It turns out that looking at them from 1 higher dimension is helpful.

TQFT in (d+1)-dimensions.

topological operators describing anyons.



d-dim. QFT of our interest



top. op. in TQFT.

sym. generator in QFT.

$\Longrightarrow$ Sym TFT. (cf. Freed, Teleman '12, '18, Gaiotto, Kulp '20, Apruzzi, Bonetti, Etxebarria, Hosseini, Schafer-Nameki '21, Kaidi, Ohmori, Zheng '22, ...)

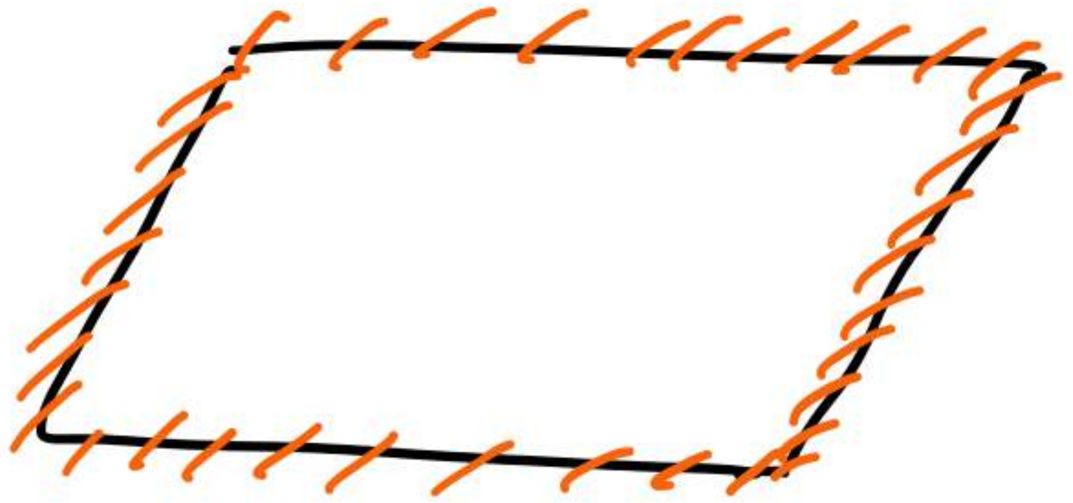
Application : Solitonic symmetry beyond homotopy

Based on my work with Shi Chen

Topological Stability and Homotopies

(Mermin '79 Rev. Mod. Phys.)

Spacetime



$$\xrightarrow{\sigma} \quad \mathcal{M} \ (= G/H\ ).$$

To find finite action/energy (density) $\int |d\sigma|^2 < \infty$, its convenient to identifiy ∞'s of $\mathbb{R}^n$:

$$\mathbb{R}^n \cup \{\infty\} \simeq S^n \xrightarrow{\sigma} \mathcal{M}.$$

$\Rightarrow$ Topoligcal solitons are classified by homotopies of the target space

$$\pi_n(\mathcal{M}).$$

$\boxed{\text{Conventional Wisdom : Solitonic Sym.} \cong \mathrm{Hom}(\pi_n(\mathcal{M}), U(1)).}$ Is this always true?

4d $\mathbb{C}P^1$ σ-model

Assume some (3+1) d quantum systems have SSB

$$SU(2) \xrightarrow{SSB} U(1).$$

The target space of the nonlinear σ-model becomes

$$\mathbb{C}P^1 \simeq SU(2)/U(1)$$

Lagrangian:

$$\mathcal{L} = \frac{1}{g^2} \left| (\partial_\mu + i a_\mu) \vec{z} \right|^2. \quad \begin{cases} \vec{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} & : \mathbb{C}^2\text{- valued scalar field with } |\vec{z}|^2 = 1 \\ a = a_\mu dx^\mu & : \text{(auxiliary) } U(1) \text{ gauge field} \end{cases}$$

This $U(1)$ gauge field a is auxiliary because its EoM can be solved as

$$a = i \vec{z}^\dagger \cdot d\vec{z}$$

Homotopy of $\mathbb{C}P^1$ ($\simeq S^2$):

$$\pi_1(\mathbb{C}P^1) \simeq 0, \quad \underline{\pi_2(\mathbb{C}P^1) \simeq \mathbb{Z}}, \quad \underline{\pi_3(\mathbb{C}P^1) \simeq \mathbb{Z}}$$

"magnetic skyrmion" "Hopfion"

"monopole"

Selection rule of Hopfions

Let's evaluate correlation functions in a compact spacetime.

$$\left\langle H_{k_1}(x_1) \quad H_{k_2}(x_2) \quad H_{k_3}(x_3) \right\rangle \neq 0 \implies k_1 + k_2 + k_3 + \cdots = 0.$$

U(1) conservation law

$$\left\langle H_{k_1}(x_1) \quad H_{k_2}(x_2) \quad V_{n,k}(L) \right\rangle \neq 0 \implies k_1 + k_2 + \cdots + k = 0 \mod 2n.$$

$\mathbb{Z}_{2n}$ conservation law.

[cf. Pontrjagin, 1941]

- Without $V(L)$, the Hopfion number conserves as if there is a $U(1)$ symmetry.
- With $V(L)$, the conservation law reduces to that of $\mathbb{Z}_2$.

Which is the symmetry group? Or, is it something else?

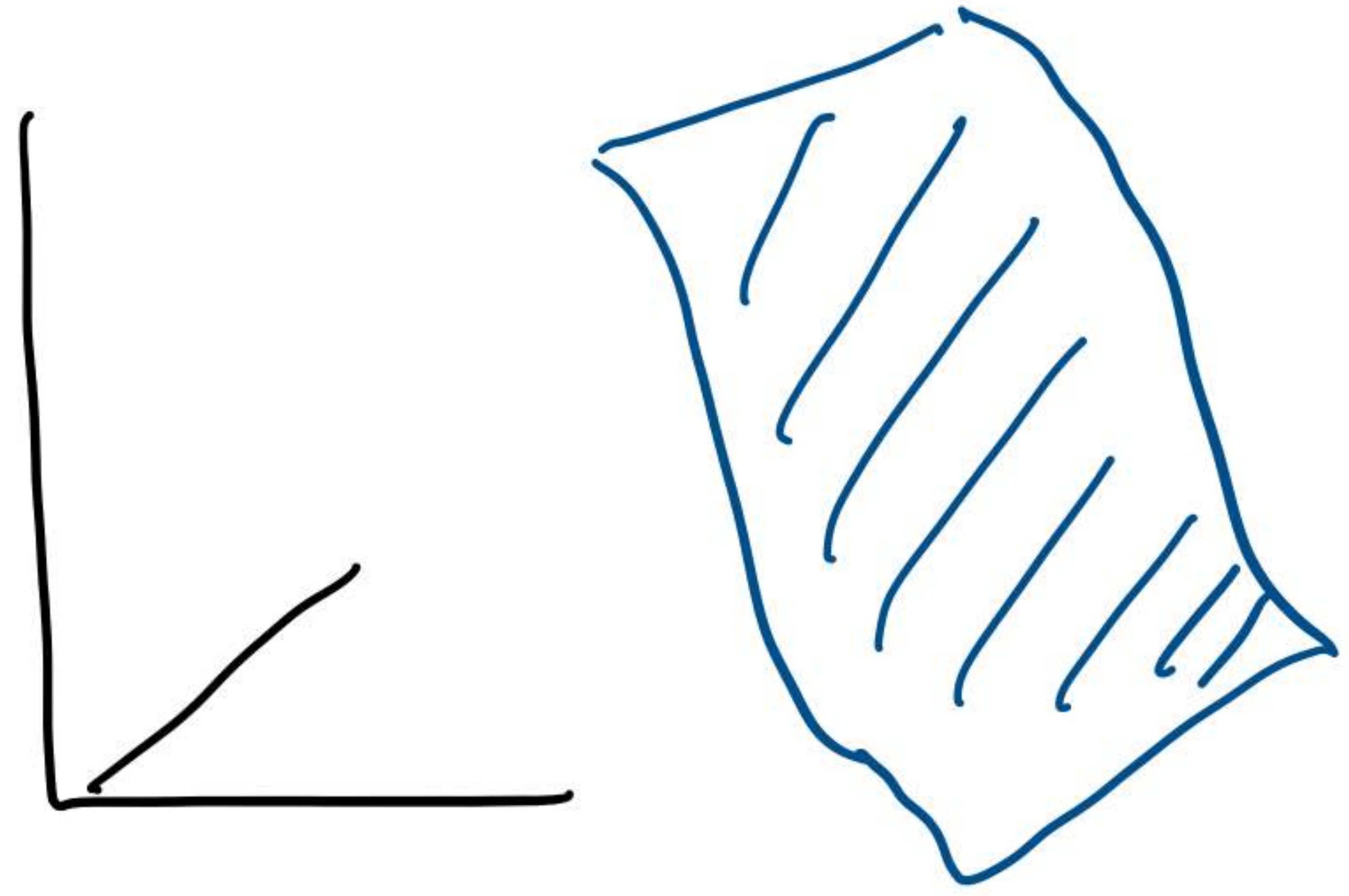
For 4d $\mathbb{C}P^1$ σ-model,

the Hopfion symmetry associated with $\pi_3(\mathbb{C}P^1) \simeq \mathbb{Z}$ is neither $U(1)$ nor $\mathbb{Z}_2$.

The correct symmetry generator is given by

$$\mathcal{H}_{\frac{\pi}{N}}(M_3) = \int \mathcal{D}b \, e^{i\frac{N}{4\pi}\int_{M_3} b\wedge db + i\frac{1}{2\pi}\int_{M_3} b\wedge da},$$

and the fusion rule is controlled by those of 3d TQFTs.



Bulk:
$(3+1)$d $\mathbb{C}P^1$ σ-model
described by $\vec{z}$ & a_μ

$(2+1)$d Chern-Simons TQFT living on M_3
described by b_μ.

invertible $\mathbb{Z}_2$ subgroup

$$\mathcal{H}_\pi(M_3) = e^{i\pi\int_{M_3}\frac{a\,da}{4\pi^2}} \in \mathbb{Z}_2 \left(\simeq \tilde{\Omega}_3^{Spin}(\mathbb{C}P^1)\right).$$

Summary

Take-home message

$$\boxed{\text{Symmetry} = \text{Topological defect operators}}$$

$\Rightarrow$ New aspects of strongly-coupled QFTs.